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Integral Relative Grothendieck Rings*

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1. INTRODUCTION

Let R be a Dedekind ring and G a finite group. An *RG-lattice* is an RG-module which is finitely generated and projective as an R-module. Let H be a fixed subgroup of G. Form the free abelian group X on the symbols $[M]$, where M ranges over a full set of representatives of the isomorphism classes of RG-lattices. Denote by Y the subgroup of X generated by all elements $[M] - [M'] - [M'']$ such that there is an RG-exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

which is split when viewed as a sequence of RH-lattices. The quotient group X/Y will be denoted by $a_R(G, H)$. The element $[M] + Y \in a_R(G, H)$ will also be denoted by $[M]$. If M, N are RG-lattices, define $M \otimes N$ to be the RG-lattice obtained by letting G act on $M \otimes_R N$ through the formula

$$g \cdot (m \otimes n) = gm \otimes gn, \quad g \in G, \quad m \in M, \quad n \in N.$$

Then putting $[M] \cdot [N] = [M \otimes N]$, we give $a_R(G, H)$ the structure of a commutative ring with unit $[R]$, where G acts trivially on R. $a_R(G, H)$ is the *Grothendieck ring of RG-lattices relative to RH-split sequences*. When $G = H$, we write $a_R(G)$ instead of $a_R(G, G)$, for brevity.

In the event that R is a field of characteristic $p > 0$, $a_R(G, H)$ has been studied extensively by Lam, Reiner, Wigner, Dress, and Conlon (See [2]–[4], [6]–[12]). In this paper, we calculate some examples in the case where R is not a field. Our examples show that the methods of the "modular" case do not generalize to the "integral" case.

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2. Groups of Prime Order

Let p be a rational prime. Denote by R the ring of p-adic integers in the field K of p-adic numbers, and let $H = \langle h \rangle$ denote a cyclic group of order p. Denote by A the RH-lattice R with trivial action of H. Let ζ be a primitive p-th root of unity over K, and denote by B the RH-lattice $R[\zeta]$, where $h \cdot x = \zeta \cdot x$ for $x \in R[\zeta]$.

(2.1) Theorem (Heller and Reiner [5]). *A and B are the only irreducible RH-lattices. The indecomposable RH-lattices are A, B, and RH. There are nonsplit RH-exact sequences*

$$0 \to A \to RH \to B \to 0,$$
$$0 \to B \to RH \to A \to 0.$$

Further,

$$\mathrm{ext}^1_{RH}(A, B) \cong \mathrm{ext}^1_{RH}(B, A) \cong A/pA \cong Z/pZ,$$

and if

$$0 \to A \to X \to B \to 0$$
$$0 \to B \to Y \to A \to 0$$

are nonsplit exact sequences, then $X \cong Y \cong RH$.

Now let t, u be positive integers. Denote by $A^{(u)}$ the RH-lattice $A \oplus \cdots \oplus A$ (u copies), etc. Then we have

$$\mathrm{ext}^1_{RH}(B^{(t)}, A^{(u)}) \cong (A/pA)_{t\times u}\,, \tag{2.2}$$

where $(A/pA)_{t\times u}$ denotes the module of all $t \times u$ matrices over A/pA. This isomorphism may be described explicitly as follows: Let

$$\Phi = 1 + h + h^2 + \cdots + h^{p-1}. \text{ Write}$$
$$B^{(t)} = B \cdot b_1 \oplus \cdots \oplus B \cdot b_t$$
$$A^{(u)} = A \cdot a_1 \oplus \cdots \oplus A \cdot a_u\,.$$

Let $RH \cdot x_1 \oplus \cdots \oplus RH \cdot x_t$ be a free RH-module with basis $x_1, \ldots, x_t$. Define $\tau\colon \sum RH \cdot x_i \to \sum B \cdot b_i$ by $\tau(x_i) = b_i$ for $i = 1, \ldots, t$. Then $\ker \tau = \sum RH \cdot \Phi \cdot x_i$, so there is an exact sequence

$$0 \longrightarrow \sum RH \cdot \Phi \cdot x_i \longrightarrow \sum RH \cdot x_i \xrightarrow{\tau} \sum B \cdot b_i \longrightarrow 0.$$

Hence there is an exact sequence

$$\hom_{RH}\left(\sum RH \cdot x_i, \sum A \cdot a_j\right) \xrightarrow{\tau^*} \hom_{RH}\left(\sum RH \cdot \Phi \cdot x_i, \sum A \cdot a_j\right)$$
$$\to \operatorname{ext}^1_{RH}\left(\sum B \cdot b_i, \sum A \cdot a_j\right) \to \operatorname{ext}^1_{RH}\left(\sum RH \cdot x_i, \sum A \cdot a_j\right).$$

Since $\sum RH \cdot x_j$ is RH-free, the last term is zero; therefore

$$\operatorname{ext}^1_{RH}\left(\sum B \cdot b_i, \sum A \cdot a_j\right) \cong \operatorname{coker} \tau^*.$$

If $F\colon \sum RH \cdot \Phi \cdot x_i \to \sum A \cdot a_j$, then the matrix corresponding to $F + \operatorname{Im} \tau^*$ under the isomorphism (2.2) is obtained by writing $F(\Phi x_i) = \sum_j \alpha_{ij} a_j$, $\alpha_{ij} \in A$, and reducing the matrix (α_{ij}) modulo pA. The extension corresponding to $F + \operatorname{Im} \tau^*$ is the lower row of

$$\begin{tikzcd}
0 \arrow[r] & \sum RH \cdot \Phi x_i \arrow[r] \arrow[d, "F"'] & \sum RH \cdot x_i \arrow[r] \arrow[d] & \sum B \cdot b_i \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \sum A \cdot a_j \arrow[r] & M \arrow[r] & \sum B \cdot b_i \arrow[r] & 0,
\end{tikzcd}$$

where the left hand square is constructed as a pushout diagram. If r is the R/pR-rank of $(\alpha_{ij} + pA)$, then it follows from [5] that

$$M \cong RH^{(r)} \oplus A^{(u-r)} \oplus B^{(t-r)}.$$

(2.3) Lemma [5]. *Let X be an RH-lattice. Then*

(i) $\operatorname{ext}^1_{RH}(RH, X) = 0$
(ii) $\operatorname{ext}^1_{RH}(X, RH) = 0$
(iii) $\operatorname{ext}^1_{RH}(A, A) = 0$
(iv) $\operatorname{ext}^1_{RH}(B, B) = 0.$

(2.4) Lemma. *Suppose that*

$$E : 0 \longrightarrow A \xrightarrow{\alpha} M \longrightarrow M/A \longrightarrow 0$$

is a nonsplit short exact sequence of RH-lattices. Then there is an RH-sublattice X of M such that

(i) $X \cong RH$
(ii) $\alpha(A) \subseteq X$
(iii) *X is an RH-direct summand of M.*

Proof. Write M/A as a direct sum of indecomposable lattices. Then there is a corresponding decomposition of $\operatorname{ext}^1_{RH}(M/A, A)$. Since E represents a nonzero element of $\operatorname{ext}^1_{RH}(M/A, A)$, it follows that for some indecomposable summand Y of M/A, the component of the class of E in $\operatorname{ext}^1_{RH}(Y, A)$ is nonzero. By (2.3), we conclude immediately that $Y \cong B$. Since the projections in the decomposition of $\operatorname{ext}^1_{RH}(M/A, A)$ are induced by the injections in the decomposition of M/A, we see that there is an RH-split exact sequence

$$0 \longrightarrow B \xrightarrow{i} M/A \longrightarrow N \longrightarrow 0$$

for some RH-lattice N such that $i^*(\text{class of } E) \neq 0$, where

$$i^* = \operatorname{ext}^1_{RH}(i, A) : \operatorname{ext}^1_{RH}(M/A, A) \to \operatorname{ext}^1_{RH}(B, A).$$

It follows that any representative of $i^*(\text{class of } E)$ is nonsplit. Using the construction of i^*(see [13]), we see that there is, therefore, a diagram of RH-lattices

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X' & \longrightarrow & B & \longrightarrow & 0 \\ & & {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & M & \longrightarrow & M/A & \longrightarrow & 0 \end{array}$$

in which the upper row is nonsplit and the right-hand square is a pullback. It is easy to verify that σ is monic and coker $\sigma \cong N$. Now, $X' \cong RH$ by (2.1), so by (2.3(ii)), $\sigma(X')$ is an RH-summand of M. Hence taking $X = \sigma(X')$, conditions (i)–(iii) are satisfied, and so the proof is complete.

We conclude this section with

(2.5) LEMMA. *Suppose a diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & RH \\ {\scriptstyle \beta}\downarrow & & \\ RH & & \end{array}$$

is given, where α, β are RH-monomorphisms with R-torsionfree cokernels. Then there are RH-homomorphisms μ, θ: $RH \to RH$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & RH \arrow[dl, shift left, "\theta"] \\ RH \arrow[ur, shift left, "\mu"] & \end{tikzcd}$$

is commutative.

Proof. Straightforward.

3. Cyclic p-Groups

In this section, G is a cyclic group of order p^n, and R is the ring of p-adic integers. Let g be a generator for G, and let $h = g^{p^{n-1}}$. Then $H = \langle h \rangle$ is the unique subgroup of order p of G. We will calculate the ring $a_R(G, H)$. The methods used can also be used in the case $G = H' \times H$, where H' is cyclic of order p^{n-1}, and H is again cyclic of order p. We omit the details of this case. The notation for RH-lattices established in §2 will be carried over without further mention.

It is not difficult to see that the irreducible RG-lattices are $X_0, \ldots, X_n$, where for $i = 0, 1, \ldots, n$, ζ_i is a primitive p^i-th root of unity, and $X_i = R[\zeta_i]$ with $g \cdot x = \zeta_i x$ for $x \in R[\zeta_i]$.

For any RH-lattice N, denote by N^G the induced lattice $RG \otimes_{RH} N$. Then the exact sequence

$$0 \to A \to RH \to B \to 0$$

gives rise to an exact sequence

$$0 \to A^G \to RG \to B^G \to 0. \tag{3.1}$$

Further, the image of A in RH is just $RH \cdot \Phi$, so it follows readily that the image of A^G in RG is $RG \cdot \Phi$.

(3.2) Proposition. *Let N be any H-trivial RG-lattice. Then*

$$\operatorname{ext}^1_{RG}(B^G, N) \cong N/pN$$

as RG-modules.

Proof. The exact sequence (3.1) induces an exact sequence

$$\begin{aligned} \hom_{RG}(RG, N) \xrightarrow{\tau^*} \hom_{RG}(RG \cdot \Phi, N) \longrightarrow \operatorname{ext}^1_{RG}(B^G, N) \\ \longrightarrow \operatorname{ext}^1_{RG}(RG, N). \end{aligned}$$

The last term is zero, hence $\operatorname{ext}^1_{RG}(B^G, N) \cong \operatorname{coker} \tau^*$. Now, an RG-homomorphism $F\colon RG \cdot \Phi \to N$ is completely determined by $F(\Phi)$. Since the annihilator ideal of Φ is $RG \cdot (h - 1)$, and $(h - 1)N = 0$ by hypothesis, $F(\Phi)$ may be given any value in N. Thus $\hom_{RG}(RG \cdot \Phi, N) \cong N$. The map τ^* is given by restriction, hence if $F\colon RG \to N$, then $\tau^*(F)(\Phi) = F(\Phi) = \Phi \cdot F(1)$. Now $F(1)$ may take any value in N, so in the isomorphism $\hom_{RG}(RG \cdot \Phi, N) \cong N$, the image of τ^* is identified with $\Phi \cdot N$. But h acts trivially on N, so $\Phi \cdot N = p \cdot N$ and hence

$$\operatorname{ext}^1_{RG}(B^G, N) \cong N/pN,$$

as claimed.

The above proof shows that for any element $n \in N$, we may construct an extension $(B^G, N; n)$ of B^G by N. Namely, we take $F(\Phi) = n$, and we construct the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & RG \cdot \Phi & \xrightarrow{\tau} & RG & \longrightarrow & B^G & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle F} & & \Big\downarrow & & \Big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & B^G & \longrightarrow & 0, \end{array}$$

where the left-hand square is a pushout. We put $(B^G, N; n) = M$. Furthermore, the two extensions $(B^G, N; n)$ and $(B^G, N; n')$ are equivalent if and only if $n - n' \in pN$. It is easy to prove

(3.3) LEMMA. *Let α by an RG-automorphism of N. Then $(B^G, N; n) \cong (B^G, N; \alpha(n))$.*

Using the fact that $B^G \cong RG/RG \cdot \Phi$, it is easy to deduce:

(3.4) LEMMA. $X_n \cong B^G$. $(X_n)_H \cong B^{(p^{n-1})}$, *where $(X_n)_H$ denotes X_n viewed as an RH-lattice by restriction of operators.*

From the remarks above, we deduce that we may give extensions of X_n by X_0 by giving an element $x_0 \in X_0$. Let us use (3.3) to show

(3.5) LEMMA. *Up to isomorphism (of RG-lattices), there is only one nonsplit extension of X_n by X_0.*

Proof. Let $x_0 \in X_0$ and consider $(X_n, X_0; x_0)$. Since R is a discrete valuation ring and $X_0 = R$, we may write $x_0 = p^k \cdot u$, where k is a nonnegative integer and u is a unit. The map $X_0 \to X_0$, given by multiplication by u^{-1}, is an RG-automorphism of X_0, so by (3.3), we have

$$(X_n, X_0; x_0) \cong (X_n, X_0; p^k).$$

If $k \geq 1$, then $p^k \in pX_0$; so, $(X_n, X_0; p^k) \cong (X_n, X_0; 0) = X_n \oplus X_0$. Hence up to isomorphism, the only nonsplit extension of X_n by X_0 is $(X_n, X_0; 1)$.

Henceforth, $(X_n, X_0; 1)$ will be denoted by (X_n, X_0).

Let $\Gamma = \langle \gamma \rangle$ be a cyclic group of order p^{n-1}. We identify Γ with G/H; thus we have an epimorphism $\varphi: G \to \Gamma$ with kernel H. φ induces an epimorphism $RG \to R\Gamma$ through which we may regard $R\Gamma$ and its ideals as RG-lattices. In particular, let E denote $R\Gamma$ viewed as RG-lattice, and V the RG-sublattice $(\gamma - 1) R\Gamma$ of E. Observe that V is the augmentation ideal of $R\Gamma$.

Since $H = \ker \varphi$, E and V are H-trivial. Further, by [5, Theorem 1.7], E is a local ring with maximal ideal $pE + V$, and E/pE is a local principal ideal ring with maximal ideal V/pV. From this, it follows easily that any element of E is congruent modulo pE to an element of the form $(\gamma - 1)^k u$, with u a unit. Hence any extension of X_n by E is equivalent to an extension of the form $(X_n, E; (\gamma - 1)^k u)$. Now, if $k \geqslant p^{n-1}$, then $(\gamma - 1)^k \in pE$, whence $(X_n, E; (\gamma - 1)^k u) \cong X_n \oplus E$. On the other hand, if $k < p^{n-1}$, then by (3.3), we deduce

$$(X_n, E; (\gamma - 1)^k u) \cong (X_n, E; (\gamma - 1)^k).$$

Hence we have shown

(3.6) LEMMA. *All isomorphism types of RG-lattices which are extensions of B^G by E occur in the list* $(X_n, E; 1), (X_n, E; (\gamma - 1)), \ldots, (X_n, E; (\gamma - 1)^{p^{n-1}})$.

Henceforth, $(X_n, E; (\gamma - 1)^k)$ will be denoted by $(X_n, E)_k$, for $0 \leqslant k \leqslant p^{n-1}$. Similarly, we denote $(X_n, V; (\gamma - 1)^k)$ by $(X_n, V)_k$, for $0 \leqslant k \leqslant p^{n-1}$.

Now we do some preliminary calculations in $a_R(G, H)$. Observe that there is an RG-exact sequence

$$0 \longrightarrow V \xrightarrow{i} E \xrightarrow{\epsilon} X_0 \longrightarrow 0, \tag{3.7}$$

where ϵ is the augmentation map and i is the inclusion. This is an RH-split sequence. It is not difficult to verify that (3.7) induces an RH-split sequence

$$0 \to (X_n, V)_r \to (X_n, E)_r \to X_0 \to 0,$$

whence

(3.8)

$$[(X_n, E)_r] = [(X_n, V)_r] + [X_0] \quad \text{in} \quad a_R(G, H), \quad \text{for} \quad r = 0, 1, \ldots, p^{n-1}.$$

We also have an RG-epimorphism $\sigma\colon E \to V$ given by $\sigma(e) = (\gamma - 1)e$, $e \in E$. Form the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r] \arrow[d, "\sigma"'] & (X_n, E)_r \arrow[d, "\delta"] \\ & V \arrow[r, "\lambda"] & M. \end{tikzcd}$$

Then λ is monic, and $\ker \delta \cong \ker \sigma \cong X_0$. Let $F\colon RG \cdot \Phi \to E$ be given by $F(\Phi) = (\gamma - 1)^r$. Then the diagram

$$\begin{tikzcd} RG \cdot \Phi \arrow[r, "\tau"] \arrow[d, "\sigma \circ F"'] & RG \arrow[d] \\ V \arrow[r, "\lambda"] & M \end{tikzcd}$$

is a pushout, where $RG \to M$ is the composite of δ with the canonical map $RG \to (X_n, E)_r$. Hence $M \cong (X_n, V; \sigma F(\Phi))$. However, $\sigma F(\Phi) = (\gamma - 1)^{r+1}$, so $M \cong (X_n, V)_{r+1}$ by the uniqueness of pushouts. Thus, there is an RG-exact sequence

$$0 \longrightarrow X_0 \xrightarrow{\alpha} (X_n, E)_r \longrightarrow (X_n, V)_{r+1} \longrightarrow 0. \tag{3.9}$$

We will show below that (3.9) is not RH-split.

Suppose now that M is any extension of an H-trivial RG-lattice N, by $X_n (\cong B^G)$. We may construct a commutative diagram with RG-exact rows

$$\begin{tikzcd} 0 \arrow[r] & RG \cdot \Phi \arrow[r] \arrow[d, "F"'] & RG \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & N \arrow[r] & M \arrow[r] & X_n \arrow[r] & 0, \end{tikzcd}$$

by the remarks following the proof of Prop. (3.2). In order to calculate the isomorphism class of the restricted lattice M_H, it follows from the discussion of the isomorphism (2.2) and the structure of $(X_n)_H$ that we must choose an R-basis $a_1, ..., a_u$ for N, write

$$F((g-1)^i \Phi) = \sum_{j=1}^{u} \alpha_{ij} a_j, \qquad \text{with} \quad \alpha_{ij} \in R (0 \leqslant i \leqslant p^{n-1} - 1),$$

and then calculate the R/pR-rank s of the matrix $(\alpha_{ij} + pR)$. Once this is done, we will have

$$M_H \cong RH^{(s)} \oplus A^{(u-s)} \oplus B^{(p^{n-1}-s)}.$$

Let us apply this procedure to the modules $M = (X_n, E)_r$, $(X_n, V)_r$ and (X_n, X_0). The module E has R-basis $\{(\gamma - 1)^i : 0 \leqslant i \leqslant p^{n-1} - 1\}$, and $(X_n, E)_r$ is determined by the homomorphism $F: RG \cdot \Phi \to E$ given by $F(\Phi) = (\gamma - 1)^r$. Thus $F((g-1)^i \cdot \Phi) = (g-1)^i \cdot (\gamma - 1)^r = (\gamma - 1)^{r+i}$. Since $(\gamma - 1)^{p^{n-1}} \in pE$, the matrix to be considered is the $p^{n-1} \times p^{n-1}$ matrix

$$\begin{bmatrix} 000 & \cdots & 010 & \cdots & 0 \\ 000 & \cdots & 001 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & . \\ 000 & \cdots & 000 & \cdots & 1 \\ 000 & \cdots & 000 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & . \\ 000 & \cdots & 000 & \cdots & 0 \end{bmatrix}, \tag{3.10}$$

where the first nonzero entry of the first row occurs in the $(r+1)^{\text{st}}$ column. Clearly the rank of this matrix is $p^{n-1}-r$, hence

(3.11) PROPOSITION. $((X_n, E)_r)_H \cong RH^{(p^{n-1}-r)} \oplus A^{(r)} \oplus B^{(r)}$, *for* $0 \leqslant r \leqslant p^{n-1}$.

Similarly, one establishes

(3.12) PROPOSITION. $((X_n, V)_r)_H \cong RH^{(p^{n-1}-r)} \oplus A^{(r-1)} \oplus B^{(r)}$, *for* $1 \leqslant r \leqslant p^{n-1}$.

(3.13) COROLLARY. *The sequence* (3.9) *is not RH-split, for* $0 \leqslant r \leqslant p^{n-1}-1$.

Proof. By (3.11), (3.12) and the Krull–Schmidt Theorem for RH-lattices [15].

Using the method above, one may also establish

(3.14) PROPOSITION. $(X_n, X_0)_H \cong RH \oplus B^{(p^{n-1}-1)}$;

and hence

(3.15) COROLLARY. *The sequence*

$$0 \longrightarrow X_0 \xrightarrow{\beta} (X_n, X_0) \longrightarrow X_n \longrightarrow 0$$

is not RH-split.

Now, by the argument in [8, Lemma (3.1)], one may show

(3.16) LEMMA. *Let R be a Dedekind ring, $G \supseteq H$ finite groups. Let M, N, L be RG-lattices and suppose that there exists a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\beta"'] & N \arrow[dl, shift left, "\mu"] \\ L \arrow[ur, shift left, "\theta"] & \end{tikzcd}$$

such that

(i) α, β *are RG-monomorphisms,*

(ii) *coker* α, *coker* β *are RG-lattices,*

(iii) μ, θ *are RH-homomorphisms.*

Then

$$[N] - [\mathit{coker}\ \alpha] = [L] - [\mathit{coker}\ \beta] \qquad \text{in} \quad a_R(G, H).$$

(3.17) Corollary. *Let R, G, H, M, N, L, α and β be as in* (3.16). *Suppose there are RH-submodules X of N_H, Y of L_H such that*

(i) *X is an RH-summand of N_H,*

(ii) *Y is an RH-summand of L_H,*

(iii) $\operatorname{Im} \alpha \subseteq X$, $\operatorname{Im} \beta \subseteq Y$,

(iv) *There are RH-homomorphisms $\mu_1 : X \to Y$, $\theta_1 : Y \to X$ with* $\mu_1 \alpha = \beta$, $\theta_1 \beta = \alpha$.

Then

$$[N] - [\mathit{coker}\ \alpha] = [L] - [\mathit{coker}\ \beta] \quad \text{in} \quad a_R(G, H).$$

Now let $0 \leqslant r \leqslant p^{n-1} - 1$. By (2.4), (2.5), (3.13), and (3.15), we may find a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, "\alpha"] \arrow[d, "\beta"'] & Y \subseteq (X_n, E)_r \arrow[dl, shift left, "\mu_1"] \\ Z \subseteq (X_n, X_0), \arrow[ur, shift left, "\theta_1"] & \end{tikzcd}$$

where θ_1, μ_1 are RH-isomorphisms, $Z \cong Y \cong RH$, Z is an RH-summand of (X_n, X_0), and Y is an RH-summand of $(X_n, E)_r$. By (3.17), we conclude

(3.18) Proposition. *For $0 \leqslant r \leqslant p^{n-1} - 1$,*

$$[(X_n, X_0)] - [X_n] = [(X_n, E)_r] - [(X_n, V)_{r+1}] \quad \textit{in} \quad a_R(G, H).$$

(3.19) Corollary. $[(X_n, E)_{p^{n-1}-1}] = [(X_n, X_0)] + [V]$ *in* $a_R(G, H)$.

Comparing the expressions for $[(X_n, V)_r]$ obtained from (3.8) and (3.18), we obtain

(3.20) Corollary. $[(X_n, E)_{r-1}] = [(X_n, E)_r] - [X_0] - [X_n] + [(X_n, X_0)]$ *in $a_R(G, H)$, for $1 \leqslant r \leqslant p^{n-1}$.*

Using (3.20) repeatedly, along with (3.19), we obtain

(3.21) Proposition. *The subgroup of $a_R(G, H)$ generated by $[X_0], \ldots, [X_n]$, $[(X_n, E)_0], \ldots, [(X_n, E)_{p^{n-1}}]$ is also generated by $[X_0], \ldots, [X_n]$, $[(X_n, X_0)]$.*

Proof. We need only note that by (2.3(iii)), we have

$$[V] = [X_1] + \cdots + [X_{n-1}] \quad \text{in} \quad a_R(G, H).$$

Now we have two tasks remaining. Namely, we will show that $a_R(G, H)$ is generated by all $[X_i]$ and all $[(X_n, E)_k]$. Then it will follow from (3.21) that $a_R(G, H)$ is generated by $[X_0],..., [X_n]$, $[(X_n, X_0)]$. Finally, we will show that this latter set of generators is independent.

We have seen that each irreducible RG-lattice is either H-trivial, or has the property that $Y_H \cong B^{(p^{n-1})}$. Let us say that an RG-lattice M is of *type* 1 if $M_H \cong A^{(u)}$ for some u, and of *type* 2 if $M_H \cong B^{(t)}$ for some t. We have just observed that each irreducible lattice belongs to one of the types. There exist lattices which are of neither type.

(3.22) LEMMA. *If M is of type 1 or of type 2, then in $a_R(G, H)$, $[M]$ equals the sum of its composition factors.*

Proof. Clear from (2.3).

(3.23) PROPOSITION. *Let U be the subgroup of $a_R(G, H)$ generated by all $[X_i]$ and all $[(X_n, E)_k]$. Then $U = a_R(G, H)$.*

Proof. Let M be an RG-lattice; we show that $[M] \in U$ by induction on the R-rank of M. Clearly if rank $M = 0$ or if M is of type 1 or 2, then $[M] \in U$.

Hence we may assume that M is of neither type. We may then find an exact sequence

$$0 \longrightarrow I \xrightarrow{j} M \xrightarrow{k} N \longrightarrow 0 \tag{3.24}$$

of RG-lattices, with I irreducible of type 1. Also, N admits an R-pure irreducible RG-lattice of type 2. If (3.24) is RH-split, then $[M] = [I] + [N]$ belongs to U by the induction hypothesis. If (3.24) is not RH-split, we must use a much more difficult argument. Let $N' = \{n \in N \mid \Phi n = 0\}$ be the maximal type 2 sublattice of N. Then there is an exact sequence

$$0 \longrightarrow N' \xrightarrow{i} N \longrightarrow W \longrightarrow 0,$$

where W is an H-trivial RG-lattice. We form the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \xrightarrow{j'} & P & \longrightarrow & N' & \longrightarrow & 0 \\
 & & {\scriptstyle id}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & I & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0,
\end{array}$$

where the right-hand square is a pullback. Then σ is monic, and coker $\sigma \cong W$. Fix an isomorphism

$$\lambda: P_H \cong RH^{(u)} \oplus A^{(v)} \oplus B^{(w)}.$$

Since $\operatorname{Im} j'$ is H-trivial, we have $\lambda(\operatorname{Im} j') \subseteq RH^{(u)} \oplus A^{(v)}$. Thus $\operatorname{Im} j'$ is contained in an RH-summand Y of P, where $Y \cong RH^{(u)} \oplus A^{(v)}$. Let $f\colon P_H \to Y$ be the canonical RH-projection onto the summand Y. Form the diagram of RH-modules

$$\begin{tikzcd} 0 \arrow[r] & P_H \arrow[r, "\sigma"] \arrow[d, "f"'] & M_H \arrow[r] \arrow[d, "f'"] & W_H \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\sigma'"] & M' \arrow[r] & W_H \arrow[r] & 0, \end{tikzcd}$$

where the left-hand square is a pushout diagram. Since $Y \cong RH^{(u)} \oplus A^{(v)}$ and W is H-trivial, it follows from (2.3) that there is an RH-homomorphism $\mu\colon M' \to Y$ with $\mu\sigma' = Id_Y$. Let $\theta\colon Y \to P_H$ be the inclusion, so that $f\theta = Id_Y$. Then $\mu f'\sigma\theta = \mu\sigma' f\theta = Id_Y \circ Id_Y = Id_Y$. Thus, $\sigma\theta$ is split by $\mu f'$; so $\sigma(Y)$ is an RH-summand of M_H. Now, $\sigma|_Y : Y \to \sigma(Y)$ has an inverse τ, and hence we have a commutative diagram

$$\begin{tikzcd} I \arrow[r, "j'"] \arrow[d] & Y \arrow[dl, shift right, "\sigma"'] \\ \sigma(Y). \arrow[ur, shift right, "\tau"'] & \end{tikzcd}$$

Since Y is an RH-summand of P_H and $\sigma(Y)$ is an RH-summand of M_H, we may apply (3.17) to conclude

$$[M] = [P] - [N'] + [N] \qquad \text{in} \quad a_R(G, H).$$

Now, the R-ranks of N and N' are smaller than that of M, hence $[N']$, $[N] \in U$. Thus it suffices to show $[P] \in U$.

In order to deal with P, we must repeat the entire procedure with some slight variation. Thus, using P in place of M, and an R-pure, irreducible RG-sublattice J of type 2 in P in place of I, we work as before. We obtain

$$[P] = [P'] - [I] + [P/J] \qquad \text{in} \quad a_R(G, H),$$

where P' is such that there is a short exact sequence

$$0 \to J \to P' \to I \to 0.$$

Let $P'' = \{x \in P\colon (h - 1)x = 0\}$. Then $P'' \cong I$ and $P'/P'' \cong J$. Thus, we have an exact sequence

$$0 \to I \to P' \to J \to 0.$$

Since E is the representation of G on the cosets of H, we have $E \cong A^G$. As A^G is the maximal type 1 sublattice of RG, and I is an R-pure type 1 sublattice of RG, we have an embedding $s: I \to E$ with R-pure image. Form the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & P' & \longrightarrow & J & \longrightarrow & 0 \\ & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & J & \longrightarrow & 0, \end{array}$$

where the left-hand square is a pushout. Since I, E are of type 1, s is RH-split. It is easy to show that an H-splitting of s induces an H-splitting of s'. Also, coker $s' \cong$ coker s. Thus, in $a_R(G, H)$, we have

$$[P'] = [Q] - [\text{coker } s].$$

Now, coker s is of type 1, so [coker s] $\in U$ by (3.22). Hence it suffices to show $[Q] \in U$. But J is irreducible of type 2; so, $J \cong X_n$. Thus, Q is an extension of X_n by E, and hence $Q \cong (X_n, E)_k$ for some k. Thus, $[Q] \in U$, and the proof of (3.23) is complete.

As we have noted before, (3.23) implies that $a_R(G, H)$ is generated by $[X_0],..., [X]_n$, $[(X_n, X_0)]$. Let us now show that these generators are independent. Suppose we have a relation

$$\sum_{i=0}^{n} x_i[X_i] + y \cdot [(X_n, X_0)] = 0 \tag{3.25}$$

in $a_R(G, H)$, where x_i, y are integers. Then this relation also holds in the Grothendieck ring $K^0(RG) = a_R(G, 1)$. But $[(X_n, X_0)] = [X_n] + [X_0]$ in $K^0(RG)$. Since $K^0(RG)$ is free abelian on $[X_0],..., [X_n]$, it follows that $x_0 = -y = x_n$, $x_i = 0$ for $i = 1,..., n - 1$. Mapping to $a_R(H)$ by the restriction map, we get $x_0[A] + x_n \cdot p^{n-1}[B] + y. ([RH] + (p^{n-1} - 1)[B]) = 0$, using (3.14). Since $a_R(H)$ is free abelian with basis $[A]$, $[B]$, $[RH]$, it follows that $x_0 = y = x_n = 0$. Hence all coefficients in (3.25) are zero, and thus, the given generators are independent. Our proof shows also that the map

$$a_R(G, H) \to a_R(G, 1) \oplus a_R(H)$$

given by $[M] \mapsto ([M], [M_H])$ is a one-to-one mapping of rings. By [14] and [16], $a_R(G, 1) \oplus a_R(H)$ has no nonzero nilpotent elements. Hence we have shown

(3.26) THEOREM. *Let R be the ring of p-adic integers, let G be a cyclic group of order p^n and let H be the unique subgroup of order p in G. Then $a_R(G, H)$ is free as an additive group, with rank $n + 2$. As a ring, $a_R(G, H)$ has no nonzero nilpotent elements.*

Let us observe that (3.26) stands in contrast to the results in the modular case. Indeed, setting $\bar{R} = R/pR$, we have

(3.27) THEOREM. *The restriction map res*: $a_{\bar{R}}(G, H) \to a_{\bar{R}}(H)$ *is a ring isomorphism.*

Proof. See [10, Theorem 2.1].

Now, $a_{\bar{R}}(H)$ is a free abelian group of rank p (see [14]). Thus the rank of $a_{\bar{R}}(G, H)$ depends only on p. As we have just seen, the rank of $a_R(G, H)$ depends only on n. This is indeed a curious anomaly.

4. Elementary Abelian p-Groups

In this section, p is a prime, R is the ring of p-adic integers in the p-adic field K. Let G be an elementary abelian p-group of order p^n. Thus, G has generators $a_1, \ldots, a_n$ with relations $[a_i, a_j] = 1 = a_i^p$ for $i, j = 1, \ldots, n$. Let $H = \langle a_1 \rangle$. We will calculate $a_R(G, H)$.

Denote by ζ a primitive p-th root of unity over K. From [1], we deduce that KG is the direct sum of the trivial G-module K, together with $(p^n - 1)/(p - 1)$ copies of $K[\zeta]$. The generators a_i act on $K[\zeta]$ as various powers of ζ; thus RG acts through the discrete valuation ring $R[\zeta]$. As $R[\zeta]$ is a principal ideal domain, it follows that the nontrivial irreducible RG-lattices are all isomorphic to $R[\zeta]$, with the a_i acting as indicated above. Now, if k is an integer with $1 \leqslant k \leqslant p - 1$, then ζ^k is also a primitive p-th root of unity and $R[\zeta] = R[\zeta^k]$. Further, the matrix of the action of ζ on $R[\zeta]$ with respect to the R-basis $1, \zeta, \ldots, \zeta^{p-2}$ is the same as that of the action of ζ^k with respect to the R-basis $1, \zeta^k, (\zeta^k)^2, \ldots, (\zeta^k)^{p-1}$. Namely, both matrices are the companion matrix $\mathbf{C}$ of the cyclotomic polynomial $\Phi_p(X)$ over K. Hence in choosing a full set of irreducible RG-lattices, we may assume that on each of them, a_1 acts either trivially or as multiplication by ζ. As in §3, we say that an RG-lattice X is of *type* 1 if X is H-trivial, and that X is of *type* 2 if $X_H \cong B^{(t)}$, some t.

Let X be an irreducible RG-lattice of type 2. Then X affords a matrix representation

$$\begin{aligned} a_1 &\to \mathbf{C} \\ a_k &\to \mathbf{C}^{n_k}, \qquad k = 2, \ldots, n, \end{aligned}$$

where the n_k are integers lying between zero and $p - 1$. Denote by T the trivial RG-lattice. We define an RG-lattice (X, T) by requiring that it afford the matrix representation

$$\begin{aligned} a_1 &\to \mathbf{D} \\ a_k &\to \mathbf{D}^{n_k}, \qquad k = 2, \ldots, n, \end{aligned}$$

where

$$\mathbf{D} = \left(\begin{array}{c|c} 1 & 0 \quad 0 \quad \cdots \quad 1 \\ \hline 0 & \\ 0 & \\ \vdots & \mathbf{C} \\ 0 & \end{array}\right).$$

Then there are RG-exact sequences

$$0 \longrightarrow X \xrightarrow{\mu} (X, T) \longrightarrow T \longrightarrow 0. \tag{4.1}$$

$$0 \longrightarrow T \longrightarrow (X, T) \longrightarrow X \longrightarrow 0. \tag{4.2}$$

Neither sequence is RH-split, since $X_H \cong B$, $T_H \cong A$ and $(X, T)_H \cong RH$.

Imitating the proofs of (2.4) and (2.5), we may show

(4.3) LEMMA. *Suppose that*

$$0 \longrightarrow B \xrightarrow{\alpha} M \longrightarrow M/B \longrightarrow 0$$

is a nonsplit short exact sequence of RH-lattices. Then there is an RH-sublattice Y of M such that

(i) $Y \cong RH$,

(ii) $\alpha(B) \subseteq Y$,

(iii) *Y is an RH-summand of M_H.*

(4.4) LEMMA. *Suppose a diagram*

$$\begin{tikzcd} B \arrow[r, "\alpha"] \arrow[d, "\beta"'] & RH \\ RH & \end{tikzcd}$$

is given, where α, β are RH-monomorphisms with R-torsion-free cokernels. Then there is a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\alpha"] \arrow[d, "\beta"'] & RH \arrow[dl, shift left, "\theta"] \\ RH, \arrow[ur, shift left, "\mu"] & \end{tikzcd}$$

where μ, θ are RH-isomorphisms.

(4.5) LEMMA. *Let U be the subgroup of $a_R(G, H)$ generated by all $[X]$ such that X is irreducible and all $[(X, T)]$ such that X is irreducible of type 2. Then $U = a_R(G, H)$.*

Proof. Let M be an RG-lattice; we show $[M] \in U$ by induction on the R-rank of M. If M is irreducible or $M \cong (X, T)$ for some X, then $[M] \in U$. If all composition factors of M are of the same type, then in $a_R(G, H)$, $[M]$ equals the sum of its composition factors.

It remains to consider the case where M has composition factors of both types. Then there is an exact sequence

$$0 \longrightarrow X \xrightarrow{\alpha} M \longrightarrow M/X \longrightarrow 0 \tag{4.6}$$

with X irreducible of type 2. If (4.6) is RH-split, then $[M] = [X] + [M/X]$ in $a_R(G, H)$, whence $[M] \in U$ by the induction hypothesis. If (4.6) is not RH-split, then by (4.3), (4.4), and (3.17), we deduce

$$[M] = [(X, T)] - [T] + [M/X] \qquad \text{in} \quad a_R(G, H);$$

so, by the induction hypothesis, the proof is complete.

Now let X, Y be irreducible RG-lattices of type 2. Since $(X, T)_H \cong (Y, T)_H \cong RH$, and $T_H \cong A$, we may apply (2.5) and (3.16) to obtain $[(Y, T)] = [(X, T)] - [X] + [Y]$ in $a_R(G, H)$. Thus, we need only one of the $[(X, T)]$. Suppose that the irreducibles are

$$T, W_1, \ldots, W_r, X_1, \ldots, X_s,$$

where $W_1, \ldots, W_r$ are of type 1 and are not G-trivial and $X_1, \ldots, X_s$ are of type 2. Then, by (4.2.5) and the remarks above, we see that $a_R(G, H)$ is generated by $[T]$, $[W_1], \ldots, [W_r]$, $[X_1], \ldots, [X_s]$, $[(X_1, T)]$. As in §3, we can show that these are independent. We obtain

(4.7) THEOREM. *Let R denote the p-adic integers, let G be an elementary abelian p-group and let H be a subgroup of order p. Then $a_R(G, H)$ is free abelian as an additive group, with rank $(p^n - 1)/(p - 1) + 2$. As a ring, $a_R(G, H)$ has no nonzero nilpotent elements.*

Let us remark that we have produced another counter example to the methods of the modular case. Indeed, Lam and Reiner [8, Theorem 5.1] have shown

(4.8) THEOREM. *Let G be a direct product $B \times K$ of finite groups B, K*

and let $H \subseteq K$. Assume that the field F of finite characteristic is a splitting field for B and all of its subgroups. Then, there is a ring isomorphism

$$a_F(B, 1) \otimes_{\mathbf{Z}} a_F(K, H) \cong a_F(G, H),$$

where $\mathbf{Z}$ denotes the ring of rational integers.

Let $\bar{R} = R/pR$, $B = \langle a_2, ..., a_n \rangle$, $H = K = \langle a_1 \rangle$. Then (4.8) yields a ring isomorphism

$$a_{\bar{R}}(B, 1) \otimes_{\mathbf{Z}} a_{\bar{R}}(H) \cong a_{\bar{R}}(G, H).$$

However, there is no ring isomorphism or even group isomorphism between $a_R(B, 1) \otimes_{\mathbf{Z}} a_R(H)$ and $a_R(G, H)$ (since these groups have different ranks). Thus, (4.8) fails in the integral case. Finally, we note that (4.7) gives another counter example to the integral version of [10, Theorem 2.1].

5. Quaternion and Dihedral Groups

Let R denote the ring of 2-adic integers. Let G be a nonabelian group of order 8, so G is quaternion or dihedral. Let $H = [G, G]$; then $H \lhd G$ and $|H| = 2$. By methods similar to those in §3 and §4, one can show

(5.1) Theorem. *$a_R(G, H) = U \oplus V$, where U is free abelian of rank six and $|V| \leqslant 2$.*

The group V is generated by $[Y_1] - [Y_2]$, where Y_1, Y_2 are the two inequivalent RG-modules generating the unique irreducible KG-module of dimension > 1. It seems quite difficult to determine whether or not $V = 0$.

On the basis of §3–§5, we conjecture that if $R = p$-adic integers, G is a p-group, $H \lhd G$ and $|H| = p$, then $a_R(G, H)$ is free abelian of rank $e(G) + 1$, where $e(G)$ denotes the number of irreducible representations of G in the p-adic field. One may adapt the proof of (3.23) to give at least a partial proof of this conjecture.

6. Miscellany

An RG-lattice M is *(G, H)-projective* if every short exact sequence

$$0 \to N \to L \to M \to 0$$

of RG-lattices, which is RH-split is also RG-split. The subgroup $k_R(G, H)$ of $a_R(G)$ generated by all $[M]$ such that M is (G, H)-projective is an ideal, and there is a natural map

$$c\colon k_R(G, H) \to a_R(G, H)$$

given by $c([M]) = [M]$. c is the *Cartan homomorphism.* By the same proof as in [8], one may show

(6.1) THEOREM. *Let R be a complete discrete valuation ring. Then c is monic.*

From this, one can deduce

(6.2) COROLLARY. *Let R be a Dedekind ring such that the Jordan–Zassenhaus Theorem holds for RG-lattices and the residue fields of R have finite characteristics. Then* ker c *is finite.*

(6.3) COROLLARY. *Let R be a discrete valuation ring (not necessarily complete) with finite residue field. Then c is monic.*

(6.4) COROLLARY. *Let R be a complete discrete valuation ring and assume that the Jordan–Zassenhaus Theorem holds for RH-lattices. Suppose that $a_R(G, H)$ is finitely generated. Then the number $n(RH)$ of isomorphism classes of indecomposable RH-lattices is finite.*

(6.5) COROLLARY. *Let K be an algebraic number field which is a splitting field for G, and let R be the ring of algebraic integers in K. Suppose that $a_R(G, H)$ is finitely generated. Then $n(RH) < \infty$.*

On the basis of (6.4), (6.5), and the results of [9], we conjecture that $a_R(G, H)$ is finitely generated if and only if $n(RH) < \infty$.

One may also show that two other modular results do not carry over to the integral case. Namely, Lam and Reiner [6] and Dress [4] have shown that the cokernel of c is a torsion group in the modular case. In contrast, we have

(6.6) PROPOSITION. *Let $R = p$-adic integers, G a p-group, H a subgroup of order p. If $H \neq G$, then the cokernel of c has elements of infinite order.*

Also, in the modular case, it is shown in [10] that the restriction map

$$\text{res}: a_F(G, H) \to a_F(H)$$

is often monic or even an isomorphism. We have

(6.7) PROPOSITION. *Let R be a domain of characteristic zero. Let $H \neq G$. Then* res: $a_R(G, H) \to a_R(H)$ *is not monic.*

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